Berechne mit Hilfe des Levi-Civita-Tensors ϵ_{ijk} das vierfache Vektorprodukt

$$\vec{x} = (\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}). \tag{1}$$

$$\begin{aligned}
x_i &= \Big[(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d})\Big]_i \\
&= \sum_{j,k=1}^{3} \epsilon_{ijk} [\vec{a} \times \vec{b}]_j \cdot [\vec{c} \times \vec{d}]_k \\
&= \sum_{j,k=1}^{3} \epsilon_{ijk} \left(\sum_{m,n=1}^{3} \epsilon_{jmn} a_m b_n \right) \cdot \left(\sum_{r,s=1}^{3} \epsilon_{krs} c_r d_s \right) \\
&= \sum_{j,k,m,n,r,s=1}^{3} \epsilon_{ijk}\, \epsilon_{jmn}\, \epsilon_{krs}\, a_m\, b_n\, c_r\, d_s \\
&= \sum_{j,m,n,r,s=1}^{3} \epsilon_{jmn}\, a_m\, b_n\, c_r\, d_s \underbrace{\left(\sum_{k=1}^{3} \epsilon_{ijk}\, \epsilon_{krs} \right)}_{=\delta_{ir}\, \delta_{js} - \delta_{is}\, \delta_{jr}} \\
&= \sum_{j,m,n,r,s=1}^{3} \epsilon_{jmn}\, a_m\, b_n\, c_r\, d_s (\delta_{ir}\, \delta_{js} - \delta_{is}\, \delta_{jr}) \\
&= \sum_{j,m,n=1}^{3} \epsilon_{jmn}\, a_m\, b_n \left(\sum_{r,s=1}^{3} c_r\, d_s (\delta_{ir}\, \delta_{js} - \delta_{is}\, \delta_{jr}) \right) \\
&= \sum_{j,m,n=1}^{3} \epsilon_{jmn}\, a_m\, b_n \left(\sum_{r=1}^{3} (c_r\, d_j\, \delta_{ir} - c_r\, d_i\, \delta_{jr}) \right) \\
&= \sum_{j,m,n=1}^{3} \epsilon_{jmn}\, a_m\, b_n (c_i\, d_j - c_j\, d_i) \\
&= c_i \cdot \sum_{j,m,n=1}^{3} \epsilon_{jmn}\, a_m\, b_n\, d_j - d_i \cdot \sum_{j,m,n=1}^{3} \epsilon_{jmn}\, a_m\, b_n\, c_j \\
&= c_i \cdot \sum_{j=1}^{3} d_j \underbrace{\left(\sum_{m,n=1}^{3} \epsilon_{jmn}\, a_m\, b_n \right)}_{[\vec{a} \times \vec{b}]_j} - d_i \cdot \sum_{j=1}^{3} j c_j \underbrace{\left(\sum_{m,n=1}^{3} \epsilon_{jmn}\, a_m\, b_n \right)}_{[\vec{a} \times \vec{b}]_j} \\
&= c_i \cdot \underbrace{\sum_{j=1}^{3} d_j \cdot [\vec{a} \times \vec{b}]_j}_{\vec{d} \cdot [\vec{a} \times \vec{b}]} - d_i \cdot \underbrace{\sum_{j=1}^{3} c_j \cdot [\vec{a} \times \vec{b}]_j}_{\vec{c} \cdot [\vec{a} \times \vec{b}]} \\
&= c_i \cdot \left(\vec{d} \cdot [\vec{a} \times \vec{b}] \right) - d_i \cdot \left(\vec{c} \cdot [\vec{a} \times \vec{b}] \right) \qquad (2)
\end{aligned}$$